

METRICS WITH PRESCRIBED HORIZONTAL BUNDLE ON SPACES OF CURVES

MARTIN BAUER AND PHILIPP HARMS

ABSTRACT. We study metrics on the shape space of curves that induce a prescribed splitting of the tangent bundle. More specifically, we consider reparametrization invariant metrics G on the space $\text{Imm}(S^1, \mathbb{R}^2)$ of parametrized regular curves. For many metrics the tangent space $T_c \text{Imm}(S^1, \mathbb{R}^2)$ at each curve c splits into vertical and horizontal components (with respect to the projection onto the shape space $B_i(S^1, \mathbb{R}^2) = \text{Imm}(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$ of unparametrized curves and with respect to the metric G). In a previous article we characterized all metrics G such that the induced splitting coincides with the natural splitting into normal and tangential parts. In these notes we extend this analysis to characterize all metrics that induce any prescribed splitting of the tangent bundle.

1. INTRODUCTION

Let $\text{Imm}(S^1, \mathbb{R}^2)$ be the space of regular planar curves. Our center of attention lies on the shape space of unparametrized curves, which can be identified with the quotient space

$$B_i(S^1, \mathbb{R}^2) = \text{Imm}(S^1, \mathbb{R}^2)/\text{Diff}(S^1).$$

Here, $\text{Diff}(S^1)$ denotes the Lie group of all smooth diffeomorphisms on the circle, which acts smoothly on $\text{Imm}(S^1, \mathbb{R}^2)$ via composition from the right:

$$\text{Imm}(S^1, \mathbb{R}^2) \times \text{Diff}(S^1) \rightarrow \text{Imm}(S^1, \mathbb{R}^2), \quad (c, \varphi) \mapsto c \circ \varphi.$$

The quotient space $B_i(S^1, \mathbb{R}^2)$ is not a manifold, but only an orbifold with isolated singularities (see [6] for more information). A strong motivation for considering this space – and in particular Riemannian metrics thereon – comes from the field of shape analysis [8, 9, 13, 14, 15, 2]. See [3] for a recent overview on various metrics on these spaces.

Date: November 19, 2015.

2010 Mathematics Subject Classification. Primary 58D17, 58E30, 35A01.

Key words and phrases. curve matching; shape space; Sobolev type metric; reparametrization group; Riemannian shape analysis; Riemannian submersion; infinite dimensional geometry.

M. Bauer was supported by the European Research Council (ERC), within the project 306445 (Isoperimetric Inequalities and Integral Geometry) and by the FWF-project P24625 (Geometry of Shape spaces).

Given a reparametrization invariant metric G on $\text{Imm}(S^1, \mathbb{R}^2)$, we can (under certain conditions) induce a unique Riemannian metric on the quotient space $B_i(S^1, \mathbb{R}^2)$ such that the projection

$$(1) \quad \pi : \text{Imm}(S^1, \mathbb{R}^2) \rightarrow B_i(S^1, \mathbb{R}^2) := \text{Imm}(S^1, \mathbb{R}^2) / \text{Diff}(S^1)$$

is a Riemannian submersion. A detailed description of this construction is given in [5, Section 4]. For many metrics, $T\pi$ induces a splitting of the tangent bundle $T\text{Imm}(S^1, \mathbb{R}^2)$ into a vertical bundle, which is defined as the kernel of $T\pi$, and a horizontal bundle, defined as the G -orthogonal complement of the vertical bundle:

$$(2) \quad T\text{Imm}(S^1, \mathbb{R}^2) = \ker T\pi \oplus (\ker T\pi)^{\perp, G} =: \text{Ver} \oplus \text{Hor}.$$

If one can lift any curve in $B_i(S^1, \mathbb{R}^2)$ to a horizontal curve in $\text{Imm}(S^1, \mathbb{R}^2)$, then there is a one-to-one correspondence between geodesics on shape space $B_i(S^1, \mathbb{R}^2)$ and horizontal geodesics on $\text{Imm}(S^1, \mathbb{R}^2)$ [5, Section 4.8].

In [4] we described all metrics on $\text{Imm}(S^1, \mathbb{R}^2)$ such that the splitting (2) coincides with the natural splitting into components that are tangential and normal to the immersed surface. In this article we characterize all metrics that induce an arbitrary given splitting of $T\text{Imm}(S^1, \mathbb{R}^2)$, generalizing our previous result.

As an application of our result we investigate a splitting that could be used to develop efficient numerics for the horizontal geodesic equation. The splitting is the decomposition of $T_c\text{Imm}(S^1, \mathbb{R}^2)$ into deformations preserving the speed $\|\dot{c}\|$ and a suitable complement.

2. THE DECOMPOSITION THEOREM

2.1. Assumptions. Following [5], we now describe the class of metrics that we study in this article. We define all metrics via a so-called inertia operator L by the formula

$$(3) \quad G_c^L(h, k) = \int_{S^1} \langle L_c h, k \rangle ds,$$

where $ds = |c'|d\theta$ denotes integration by arc-length. We assume that L is a smooth bundle automorphism of $T\text{Imm}(S^1, \mathbb{R}^2)$ such that at every $c \in \text{Imm}(S^1, \mathbb{R}^2)$, the operator

$$L_c : T_c\text{Imm}(S^1, \mathbb{R}^2) \rightarrow T_c\text{Imm}(S^1, \mathbb{R}^2)$$

is a pseudo-differential operator of order $2l$ which is symmetric and positive with respect to the L^2 -metric on $\text{Imm}(S^1, \mathbb{R}^2)$. Moreover, we assume that L is invariant under the action of the reparametrization group $\text{Diff}(S^1)$ acting on $\text{Imm}(S^1, \mathbb{R}^2)$, i.e.,

$$L_{c \circ \varphi}(h \circ \varphi) = L_c(h) \circ \varphi \quad \text{for all } \varphi \in \text{Diff}(S^1).$$

These assumptions remain in place throughout this work. Their immediate use is as follows: being symmetric and positive, L induces a Sobolev-type

metric on the manifold of immersions through equation (3). The $\text{Diff}(S^1)$ -invariance of L implies the $\text{Diff}(S^1)$ -invariance of the metric G^L . Assuming that the decomposition in horizontal and vertical bundles exists, there is a unique metric on $B_i(S^1, \mathbb{R}^2)$ such that the projection (1) is a Riemannian submersion (see [5, Thm. 4.7]). Then the resulting geometry of shape space is mirrored by the “horizontal geometry” on the manifold of immersions.

2.2. Splitting into horizontal and vertical subbundles. By definition, the horizontal and vertical bundles are given by

$$\text{Ver}_c := \ker(T\pi) = \text{Tan}_c,$$

$$\text{Hor}_c := (\text{Ver}_c)^{\perp, G} = \{h \in T_c \text{Imm}(S^1, \mathbb{R}^2) : L_c h \in \text{Nor}_c\}.$$

Note, that $\text{Hor} \oplus \text{Ver}$ might not span all of $T\text{Imm}$ in this infinite-dimensional setting.

2.3. Constructing metrics that induce a prescribed splitting. We now state our main result.

Theorem 1 (Decomposition theorem). *Let $c \in \text{Imm}(S^1, \mathbb{R}^2)$ and let $H(c)$ be any complement of $\text{Tan}(c)$, i.e.,*

$$(4) \quad T_c \text{Imm}(S^1, \mathbb{R}^2) = \text{Tan}(c) \oplus H(c).$$

Then the following conditions on a metric G^L are equivalent:

- (a) *The tangent bundle splits into a vertical and horizontal bundle (2) and this splitting coincides with (4).*
- (b) *The inertia operator L_c admits a decomposition*

$$L_c = (P^{\text{tan}})^* \circ \tilde{L}_c \circ P^{\text{tan}} + (P^H)^* \circ \tilde{L}_c \circ P^H,$$

where $\tilde{L}_c : T_c \text{Imm}(S^1, \mathbb{R}^2) \rightarrow T_c \text{Imm}(S^1, \mathbb{R}^2)$ is an invertible pseudo-differential operator and where $$ denotes the adjoint with respect to the reparametrization-invariant L^2 -metric.*

The theorem follows directly as a special case of the following lemma by setting $K(c) = \text{Tan}(c)$.

Lemma 2. *Let $c \in \text{Imm}(S^1, \mathbb{R}^2)$ and let*

$$(5) \quad T_c \text{Imm}(S^1, \mathbb{R}^2) = H(c) \oplus K(c)$$

be a given splitting with corresponding projections P^H, P^K . Then the following conditions on a metric G^L are equivalent:

- (a) *The subspace $H(c)$ is G^L -orthogonal to $K(c)$.*
- (b) *The operator L_c has a decomposition*

$$L_c = (P^H)^* \circ \tilde{L}_c \circ P^H + (P^K)^* \circ \tilde{L}_c \circ P^K,$$

where $\tilde{L}_c : T_c \text{Imm}(S^1, \mathbb{R}^2) \rightarrow T_c \text{Imm}(S^1, \mathbb{R}^2)$ is an invertible pseudo-differential operator and where $$ denotes the adjoint with respect to the reparametrization-invariant L^2 -metric.*

Proof. Assume (a) and let $h = P^H(h) + P^K(h) := h^H + h^K$. We have

$$\begin{aligned} G_c(h^H + h^K, k^H + k^K) \\ &= G_c(h^H, k^H) + G_c(h^H, k^K) + G_c(h^K, k^H) + G_c(h^K, k^K) \\ &= G_c(h^H, k^H) + 0 + G_c(h^K, k^K), \end{aligned}$$

where the last equality follows from the orthogonality of the splitting with respect to the metric G . Now the formula for the operator L follows directly.

Conversely, assume (b). To see the orthogonality we calculate

$$\begin{aligned} G_c(h^H, k^K) &= \int_{S^1} \langle Lh^H, k^K \rangle ds = \int_{S^1} \langle 0 + (P^H)^*(\tilde{L}_c(h^H)), k^K \rangle ds \\ &= \int_{S^1} \langle (\tilde{L}_c(h^H), P^H(k^K)) \rangle ds = 0. \end{aligned} \quad \square$$

3. APPLICATIONS

3.1. The splitting into tangential and normal vector fields. In this section, we want to recover the results from [4]. Letting n denote the unit length normal vector field to the curve c , we define

$$(6) \quad \text{Nor}(c) := \{h = a.n : a \in C^\infty(S^1)\}.$$

This yields a splitting

$$(7) \quad T_c \text{Imm}(S^1, \mathbb{R}^2) = \text{Tan}(c) \oplus \text{Nor}(c)$$

with corresponding projections

$$\begin{aligned} (P^{\text{tan}})^*(h) &= P^{\text{tan}}(h) = \langle h, v \rangle v, \\ (P^{\text{nor}})^*(h) &= P^{\text{nor}}(h) = \langle h, n \rangle n. \end{aligned}$$

By Theorem 1 the splitting into horizontal and vertical bundles coincides with the above splitting (7) if and only if L can be written as

$$(8) \quad L_c = (P^{\text{tan}})^* \circ \tilde{L}_c \circ P^{\text{tan}} + (P^{\text{tan}})^* \circ \tilde{L}_c \circ P^{\text{tan}}.$$

This is the content of the main theorem of [4].

A particular class of metrics inducing this splitting are almost local metrics [7, 1]. Further examples of higher order metrics are given in [4].

3.2. The splitting into tangential and constant speed preserving vector fields. In this section we consider a different splitting, which is motivated by investigations of Riemannian metrics on the space of arc length parametrized curves [10, 11, 12]. In the following lemma we characterize all tangent vectors that preserve constant speed parametrization.

Lemma 3. *Let $c \in \text{Imm}(S^1, \mathbb{R}^2)$ be parametrized by constant speed. Then a tangent vector $h \in T_c \text{Imm}$ preserves the parametrization of c if and only if*

$$(9) \quad \langle D_s^2 h, v \rangle + \kappa \langle D_s h, n \rangle = 0$$

Here $v = \frac{c'}{|c'|}$ the unit length tangent vector field, $n = iv$ denotes the unit length normal vector field and κ the curvature of the curve.

Proof. This follows immediately from the infinitesimal action of h on the volume form ds :

$$(10) \quad D_{c,h}(ds) = D_{c,h}(|c'|d\theta) = \frac{\langle c', h' \rangle}{|c'|} d\theta = \langle v, D_s h \rangle ds .$$

Equation (9) is obtained by setting $D_s \langle v, D_s h \rangle = 0$. \square

We now describe a decomposition of $T\text{Imm}(S^1, \mathbb{R}^2)$ in a subspace that preserves constant speed and a complement with values in the tangential bundle.

Lemma 4. *For each regular curve c the tangent bundle $T_c\text{Imm}(S^1, \mathbb{R}^2)$ can be decomposed as*

$$(11) \quad T_c\text{Imm}(S^1, \mathbb{R}^2) = \text{Tan}(c) \oplus \text{Arc}^0(c) ,$$

where

$$(12) \quad \text{Tan}(c) := \{h = f.v : a \in C^\infty(S^1)\} ,$$

$$(13) \quad \text{Arc}^0(c) := \{h = a.n + b.v : D_s^2 b = D_s(a\kappa) \text{ and } b(0) = 0\} .$$

The corresponding projections onto these subspaces are given by

$$(14) \quad P^{\text{Arc}^0}(k) := k^{\text{arc}} = \langle k, n \rangle n + bv ,$$

$$(15) \quad P^{\text{Tan}}(k) := k^{\text{tan}} = \langle k, v \rangle v - bv ,$$

where b solves

$$(16) \quad D_s^2 b = D_s(\langle k, n \rangle \kappa) \text{ with } b(0) = 0 .$$

Proof. We start by showing that the projections take values in the correct spaces. Let $a = \langle k, n \rangle$. We calculate:

$$\begin{aligned} \langle D_s P^{\text{Arc}^0}(k), n \rangle &= \langle D_s(an + bv), n \rangle = \langle (D_s a + b\kappa)n + (D_s b - a\kappa)v, n \rangle \\ &= D_s a + b\kappa \end{aligned}$$

$$\begin{aligned} \langle D_s^2 P^{\text{Arc}^0}(k), v \rangle &= \langle D_s^2(an + bv), v \rangle = \langle D_s((D_s a + b\kappa)n + (D_s b - a\kappa)v), v \rangle \\ &= -\kappa(D_s a + b\kappa) + D_s^2 b - D_s(a\kappa) \end{aligned}$$

Thus we have

$$\begin{aligned} \langle D_s^2 h, v \rangle + \kappa \langle D_s h, n \rangle &= -\kappa(D_s a + b\kappa) + D_s^2 b - D_s(a\kappa) + \kappa(D_s a + b\kappa) \\ &= D_s^2 b - D_s(a\kappa) . \end{aligned}$$

This shows that $P^{\text{Arc}^0}(k)$ preserves constant speed parametrization. We choose $b(0) = 0$ to enforce uniqueness of the solutions. The mapping P^{Tan} takes values in Tan by definition. The projection property for P^{Arc^0} is clear. For P^{Tan} we have

$$(P^{\text{Tan}})^2(k) = (\text{Id} - P^{\text{Arc}^0})^2(k) = (\text{Id} - P^{\text{Arc}^0})(k) = P^{\text{Tan}}(k) .$$

Note, that this proves also that P^{Tan} (resp. P^{Arc^0}) are surjective mappings onto Tan (resp. Arc^0). As P^{Tan} and P^{Arc^0} are continuous mappings, their kernels Arc and Tan are closed subspaces. This shows that $\text{Imm}(S^1, \mathbb{R}^2)$ splits as in (11). \square

Using Theorem 1 we can now construct metrics on $\text{Imm}(S^1, \mathbb{R}^2)$ such that the horizontal bundle coincides with $\text{Arc}^0(c)$. The general formula for these metrics is given by:

$$G_c(h, k) = \int_{S^1} \langle L_c(h^{\text{tan}}), k^{\text{tan}} \rangle ds + \int_{S^1} \langle L_c(h^{\text{arc}}), k^{\text{arc}} \rangle ds$$

The simplest example, which corresponds to the identity operator L , is

$$\begin{aligned} G_c(h_1, h_2) &= \int_{S^1} \langle h_1^{\text{tan}}, h_2^{\text{tan}} \rangle ds + \int_{S^1} \langle h_1^{\text{arc}}, h_2^{\text{arc}} \rangle ds \\ &= \int_{S^1} \langle h_1 - b_1 \cdot v, h_2 - b_2 \cdot v \rangle ds + \int_{S^1} b_1 b_2 + a_1 a_2 ds \\ &= \int_{S^1} 2a_1 a_2 + \tilde{b}_1 \tilde{b}_2 - b_1 \tilde{b}_2 - \tilde{b}_1 b_2 + b_1 b_2 ds \end{aligned}$$

with

$$a_i = \langle h_i, n \rangle, \quad \tilde{b}_i = \langle h_i, v \rangle, \quad D_s^2 b_i = D_s(a_i \kappa), \quad b_i(0) = 0.$$

For $h_1 = h_2 = h$ this reads as

$$G_c(h, h) = \int_{S^1} 2a^2 + (\tilde{b} - b)^2 ds.$$

On the space of constant-speed parametrized immersions this induces the L^2 -metric studied in [10, 11].

REFERENCES

- [1] M. Bauer, Ph. Harms, and P. W. Michor. Almost local metrics on shape space of hypersurfaces in n -space. *SIAM J. Imaging Sci.*, 5(1):244–310, 2012.
- [2] Martin Bauer, Martins Bruveris, Stephen Marsland, and Peter W. Michor. Constructing reparameterization invariant metrics on spaces of plane curves. *Differential Geom. Appl.*, 34:139–165, 2014.
- [3] Martin Bauer, Martins Bruveris, and Peter W. Michor. Overview of the geometries of shape spaces and diffeomorphism groups. *J. Math. Imaging Vis.*, 50:60–97, 2014.
- [4] Martin Bauer and Philipp Harms. Metrics on spaces of surfaces where horizontality equals normality. *Differential Geometry and its Applications*, 39:166–183., 2015.
- [5] Martin Bauer, Philipp Harms, and Peter W. Michor. Sobolev metrics on shape space of surfaces. *J. Geom. Mech.*, 3(4):389–438, 2011.
- [6] V. Cervera, F. Mascaró, and P. W. Michor. The action of the diffeomorphism group on the space of immersions. *Differential Geom. Appl.*, 1(4):391–401, 1991.
- [7] P. W. Michor and D. Mumford. An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach. *Appl. Comput. Harmon. Anal.*, 23(1):74–113, 2007.

- [8] W. Mio and A. Srivastava. Elastic-string models for representation and analysis of planar shapes. In *Computer Vision and Pattern Recognition, 2004. CVPR 2004. Proceedings of the 2004 IEEE Computer Society Conference on*, volume 2, pages II–10–II–15 Vol.2, 2004.
- [9] Washington Mio, Anuj Srivastava, and Shantanu Joshi. On shape of plane elastic curves. *Int. J. Comput. Vision*, 73(3):307–324, July 2007.
- [10] Stephen C. Preston. The motion of whips and chains. *J. Differential Equations*, 251(3):504–550, 2011.
- [11] Stephen C. Preston. The geometry of whips. *Ann. Global Anal. Geom.*, 41(3):281–305, 2012.
- [12] Stephen C. Preston and Ralph Saxton. An H^1 model for inextensible strings. *Discrete Contin. Dyn. Syst.*, 33(5):2065–2083, 2013.
- [13] Jingyong Su, Sebastian Kurtek, Eric Klassen, and Anuj Srivastava. Statistical analysis of trajectories on Riemannian manifolds: bird migration, hurricane tracking and video surveillance. *Ann. Appl. Stat.*, 8(1):530–552, 2014.
- [14] Ganesh Sundaramoorthi, Andrea Mennucci, Stefano Soatto, and Anthony Yezzi. A new geometric metric in the space of curves, and applications to tracking deforming objects by prediction and filtering. *SIAM J. Imaging Sci.*, 4(1):109–145, 2011.
- [15] L. Younes, P. W. Michor, J. Shah, and D. Mumford. A metric on shape space with explicit geodesics. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 19(1):25–57, 2008.

MARTIN BAUER FAKULTÄT FÜR MATHEMATIK UND GEOINFORMATION, TU WIEN

PHILIPP HARMS: DEPARTMENT OF MATHEMATICS, ETH ZÜRICH

E-mail address: bauer.martin@univie.ac.at

E-mail address: philipp.harms@math.ethz.ch